# Lecture 2: Probability

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MCB1007 Introduction to Probability and Statistics
İstanbul Kültür University

# Outline

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## Example 1

What is probability of drawing an ace from an ordinary deck of 52 playing cards?

**Solution.** Since there are $n = 4$ aces among the $N = 52$ cards, the probability of drawing an ace is $\frac{4}{52} = \frac{1}{13}$.

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If after $N$ repetitions of an experiment, where $N$ is very large, an event is observed to occur in $n$ of these, then the probability of the event is $n/N$. This is also called the empirical probability of the event.

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Both the classical and frequency approaches have serious drawbacks, the first because the words *equally likely* are vague and the second because the *large number* involved is vague. Because of these difficulties, mathematicians have been led to an *axiomatic approach* to probability.

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## Example 3

If we toss a die, one sample space, or set of all possible outcomes, is given by $\{1, 2, 3, 4, 5, 6\}$ while another is $\{odd, even\}$.

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## Example 4

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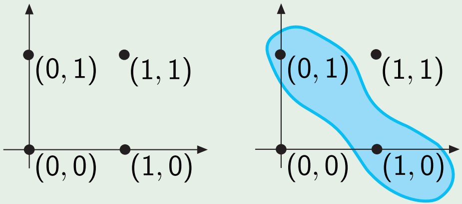
If we toss a coin twice, the event that only one head comes up is the subset of the sample space $\mathcal{S} = \{HH, HT, TH, TT\}$ that consists of points $HT$ and $TH$. Let we use 0 to represent tails and 1 to represent heads, the sample space can be portrayed by points as in the figure where, for example, $(0, 1)$ represents tails on first toss and heads on second toss, i.e., $TH$.



## Example 5

If someone takes three shots at a target and we care only whether each shot is a hit or a miss, describe a suitable sample space, the elements of the sample space that constitute event $M$ that the person will miss the target three times in a row, and the elements of event $N$ that the person will hit the target once and miss it twice.

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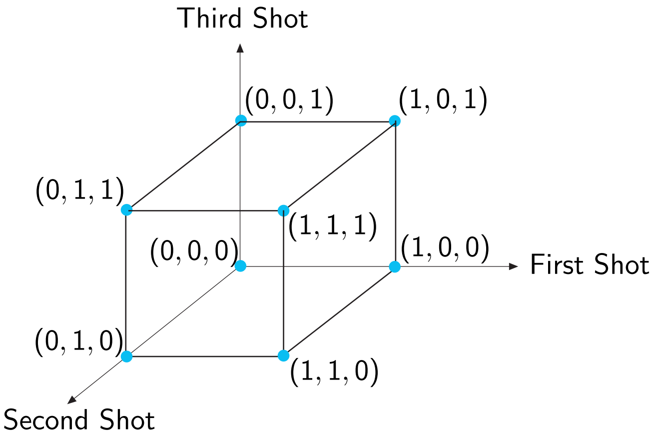
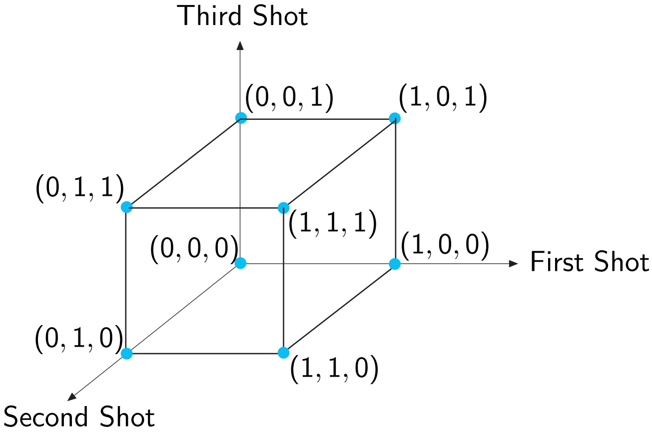
**Solution.** If we let 0 and 1 represent a miss and a hit, respectively, the eight possibilities $(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,0),(1,0,1),(0,1,1)$ and $(1,1,1)$ may be displayed as in the figure.


Third Shot
(0, 0, 1)
(1, 0, 1)
(0, 1, 1)
(1, 1, 1)
(0, 0, 0)
(1, 0, 0)
First Shot
(0, 1, 0)
(1, 1, 0)
Second Shot




Thus, it can be seen that

$$M = \{(0,0,0)\}$$

and

$$N = \{(1,0,0),(0,1,0),(0,0,1)\}.$$

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If the sets corresponding to events $A$ and $B$ are disjoint, i.e., $A \cap B = \emptyset$, we often say that the events are *mutually exclusive*. This means that they cannot both occur. We say that a collection of events $A_1, A_2, \cdots, A_n$ is mutually exclusive if every pair in the collection is mutually exclusive.

# Outline

# The Probability of an Event

To formulate the postulates of probability, we shall follow the practice of denoting events by means of capital letters, and we shall write the probability of event $A$ as $P(A)$, the probability of event $B$ as $P(B)$, and so forth. The following postulates of probability apply only to discrete sample spaces, $\mathcal{S}$.

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**P2** $P(\mathcal{S}) = 1$.

**P3** If $A_1, A_2, A_3, \cdots$, is a finite or infinite sequence of mutually exclusive events of $\mathcal{S}$, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots .$$

**Theorem 6**

*If $A$ is an event in a discrete sample space $\mathcal{S}$, then $P(A)$ equals the sum of the probabilities of the individual outcomes comprising $A$.*

## Example 7

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**Solution.** The sample space is $\mathcal{S} = \{HH, HT, TH, TT\}$. Since we assume that the coin is balanced, these outcomes are equally likely and we assign to each sample point the probability $\frac{1}{4}$. Letting $A$ denote the event that we will get at least one head, we get $A = \{HH, HT, TH\}$ and

$$\begin{aligned} P(A) &= P(HH) + P(HT) + P(TH) \\ &= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}. \end{aligned}$$

## Example 8

A die is loaded in such a way that each odd number is twice as likely to occur as each even number. Find $P(G)$, where $G$ is the event that a number greater than 3 occurs on a single roll of the die.

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**Solution.** The sample space is $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$. Hence, if we assign probability $w$ to each even number and probability $2w$ to each odd number, we find that

$$2w + w + 2w + w + 2w + w = 9w = 1$$

in accordance with Postulate 2. It follows that $w = \frac{1}{9}$ and

$$P(G) = \frac{1}{9} + \frac{2}{9} + \frac{1}{9} = \frac{4}{9}.$$

## Example 9

If, for a given experiment, $O_1, O_2, O_3, \cdots$ is an infinite sequence of outcomes, verify that

$$P(O_i) = \left(\frac{1}{2}\right)^i \text{ for } i = 1, 2, 3, \cdots$$

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**Solution.** Since the probabilities are all positive, it remains to be shown that $P(\mathcal{S}) = 1$. Getting

$$P(\mathcal{S}) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

and making use of the formula for the sum of the terms of an infinite geometric progression, we find that

$$P(\mathcal{S}) = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

## Theorem 10

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A five-card poker hand dealt from a deck of 52 playing cards is said to be a full house if it consists of three of a hand and a pair. If all the five-card hands are equally likely, what is the probability of being dealt a full house?

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**Solution.**

$$P(A) = \frac{n}{N} = \frac{\binom{4}{3}\binom{4}{2} \cdot 13 \cdot 12}{\binom{52}{5}} = 0.0014$$

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# Some Rules of Probability

**Theorem 12**

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**Proof.**

Since $A$ and $A'$ are mutually exclusive and $A \cup A' = \mathcal{S}$, we have

$$\begin{aligned} 1 &= P(\mathcal{S}) && \text{(by Postulate 2)} \\ &= P(A \cup A') \\ &= P(A) + P(A') && \text{(by Postulate 3)} \end{aligned}$$

and it follows that $P(A') = 1 - P(A)$. □

**Theorem 13**

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Since $\mathcal{S}$ and $\emptyset$ are mutually exclusive and $\mathcal{S} \cup \emptyset = \mathcal{S}$, it follows that

$$\begin{aligned} P(\mathcal{S}) &= P(\mathcal{S} \cup \emptyset) \\ &= P(\mathcal{S}) + P(\emptyset) \qquad \text{(by Postulate 3)} \end{aligned}$$

and, hence, $P(\emptyset) = 0$. □

**Theorem 14**

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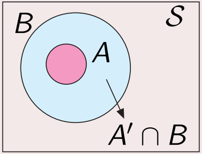
## Proof.

Since $A \subset B$, we can write $B = A \cup (A' \cap B)$.




Then, since $A$ and $A' \cap B$ are mutually exclusive, we get

$$\begin{aligned} P(B) &= P(A \cup (A' \cap B)) \\ &= P(A) + P(A' \cap B) \qquad \text{(by Postulate 3)} \\ &\geq P(A) \qquad \text{(by Postulate 1)} \end{aligned}$$

**Theorem 15**

$0 \leq P(A) \leq 1$ *for any event A.*

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**Proof.**

Using Theorem 14 and the fact that $\emptyset \subset A \subset \mathcal{S}$ for any event $A$ in $\mathcal{S}$, we have

$$P(\emptyset) \leq P(A) \leq P(\mathcal{S}).$$

Then, $P(\emptyset) = 0$ and $P(\mathcal{S}) = 1$ leads to the result that

$$0 \leq P(A) \leq 1$$

□

**Theorem 16**

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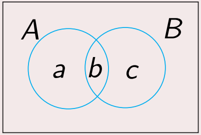
## Proof.

Assigning the probabilities $a$, $b$, and $c$ to the mutually exclusive events $A \cap B$, $A \cap B'$, and $A' \cap B$ as in the figure we find that




$$\begin{aligned} P(A \cup B) &= a + b + c \\ &= (a + b) + (c + b) - b \\ &= P(A) + P(B) - P(A \cap B). \end{aligned}$$

□

## Example 17

In a large metropolitan area, the probabilities are 0.86, 0.35, and 0.29 that a family (randomly chosen for a sample survey) owns a LCDTV set, a HDTV set or both kinds of sets. What is the probability that a family owns either or both kinds of sets?

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**Solution.** Let $A = \{\text{a family owns LCDTV}\}$, $B = \{\text{a family owns HDTV}\}$. Since $P(A) = 0.86$, $P(B) = 0.35$, and $P(A \cap B) = 0.29$, thus

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= 0.86 + 0.35 - 0.29 \\ &= 0.92. \end{aligned}$$

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**Solution.** Let $A = \{\text{a Spade is drawn}\}$, $B = \{\text{a King is drawn}\}$. Since

$$P(A) = \frac{13}{52},\ P(B) = \frac{4}{52} \text{ and } P(A \cap B) = \frac{1}{52}$$

we have

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} \\ &= \frac{16}{52} = \frac{4}{13}. \end{aligned}$$

**Theorem 19**

*If $A$, $B$ and $C$ are any three events in a sample space $\mathcal{S}$, then*

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

# Outline

# Conditional Probability

Let $A$ and $B$ be two events such that $P(A) > 0$. Denote by $P(B|A)$ the probability of $B$ given that $A$ has occurred. Since $A$ is known to have occurred, it becomes the new sample space replacing the original $\mathcal{S}$.

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**Definition 20**

If $A$ and $B$ are two any events in a sample space $\mathcal{S}$ and $P(A) > 0$, the *conditional probability* of $B$ given $A$ is

$$P(B|A) = \frac{P(A \cap B)}{P(A)}.$$

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**Solution. (a)** Let $B$ denote the event $\{\text{less than } 4\} = \{1, 2, 3\}$. Since $B$ is the union of the events 1, 2, or 3 turning up, we see that

$$P(B) = P(1) + P(2) + P(3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

assuming equal probabilities for the sample points.

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assuming equal probabilities for the sample points.
**(b)** Letting $A$ be the event $\{\text{odd number}\} = \{1, 3, 5\}$, we see that $P(A) = \frac{3}{6}$. Also $P(A \cap B) = \frac{2}{6}$. Then

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}.$$

## Example 21

Find the probability that a single toss of a die will result in a number less than 4 if **(a)** no other information is given and **(b)** it is given that the toss resulted in an odd number.

**Solution.** **(a)** Let $B$ denote the event $\{\text{less than } 4\} = \{1, 2, 3\}$. Since $B$ is the union of the events 1, 2, or 3 turning up, we see that

$$P(B) = P(1) + P(2) + P(3) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

assuming equal probabilities for the sample points.
**(b)** Letting $A$ be the event $\{\text{odd number}\} = \{1, 3, 5\}$, we see that $P(A) = \frac{3}{6}$. Also $P(A \cap B) = \frac{2}{6}$. Then

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{2}{6}}{\frac{3}{6}} = \frac{2}{3}.$$

Hence, the added knowledge that the toss results in an odd number raises the probability from $1/2$ to $2/3$.

## Example 22

The probability of a flight departing on time is $P(D) = 0.83$. The probability of a flight arriving on time is $P(A) = 0.82$. We also know that the probability that a flight both departs and arrives on time is $P(D \cap A) = 0.78$. **(a)** What is the probability of a flight arriving on time if we know it departed on time? **(b)** What is the probability of a flight departed on time if we know it arrived on time?

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**Solution.** **(a)** $P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94$.

## Example 22

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**Solution. (a)** $P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{0.78}{0.83} = 0.94$.
**(b)** $P(D|A) = \frac{P(A \cap D)}{P(A)} = \frac{0.78}{0.82} = 0.95$.

**Theorem 23 (Multiplication Rule)**

*If $A$ and $B$ are any two events in a sample space $\mathcal{S}$ and $P(B) \neq 0$, then*

$$P(A \cap B) = P(B) \cdot P(A|B).$$

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## Example 24

If we randomly pick 2 television tubes in succession from a shipment of 240 television tubes of which 15 are defective, what is the probability that they will both be defective?

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## Example 24

If we randomly pick 2 television tubes in succession from a shipment of 240 television tubes of which 15 are defective, what is the probability that they will both be defective?

**Solution.** Let $A = \{\text{the first tube is defective}\}$ and $B = \{\text{the second tube is defective}\}$. Then

$$P(A) = \frac{15}{240} \text{ and } P(B|A) = \frac{14}{239}.$$

Thus, the probability that both tubes will be defective is

$$P(A) \cdot P(B|A) = \frac{15}{240} \cdot \frac{14}{239} = \frac{7}{1,912}.$$

This assumes that we are sampling *without replacement*; that is, the first tube is not replaced before the second tube is selected.

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### Example 25

Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards is we sample **(a)** without replacement **(b)** with replacement. **(c)** It is given that the first card was an ace. What is the probability that the second card (without replacement) is also an ace?

This assumes that we are sampling *without replacement*; that is, the first tube is not replaced before the second tube is selected.

**Example 25**

Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards is we sample **(a)** without replacement **(b)** with replacement. **(c)** It is given that the first card was an ace. What is the probability that the second card (without replacement) is also an ace?

**Solution. (a)** $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$.

This assumes that we are sampling *without replacement*; that is, the first tube is not replaced before the second tube is selected.

## Example 25

Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards is we sample **(a)** without replacement **(b)** with replacement. **(c)** It is given that the first card was an ace. What is the probability that the second card (without replacement) is also an ace?

**Solution.** **(a)** $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$.
**(b)** $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$.

This assumes that we are sampling *without replacement*; that is, the first tube is not replaced before the second tube is selected.

## Example 25

Find the probabilities of randomly drawing two aces in succession from an ordinary deck of 52 playing cards is we sample **(a)** without replacement **(b)** with replacement. **(c)** It is given that the first card was an ace. What is the probability that the second card (without replacement) is also an ace?

**Solution.** **(a)** $\frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$.
**(b)** $\frac{4}{52} \cdot \frac{4}{52} = \frac{1}{169}$.
**(c)** $\frac{\frac{4}{52} \cdot \frac{3}{51}}{\frac{4}{52}} = \frac{3}{51}$.

**Theorem 26**

*If $A$, $B$, and $C$ are any three events in a sample space $\mathcal{S}$ such that $P(A \cap B) \neq 0$, then*

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B).$$

**Theorem 26**

*If $A$, $B$, and $C$ are any three events in a sample space $\mathcal{S}$ such that $P(A \cap B) \neq 0$, then*

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B).$$

**Proof.**

$$\begin{aligned} P(A \cap B \cap C) &= P[(A \cap B) \cap C] \\ &= P(A \cap B) \cdot P(C|A \cap B) \\ &= P(A) \cdot P(B|A) \cdot P(C|A \cap B). \end{aligned}$$

□

## Example 27

A box of fuses contains 20 fuses, of which 5 are defective. If 3 of the fuses are selected at random and removed from the box in succession without replacement, what is the probability that all 3 fuses are defective?

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**Solution.** If $A$ is the event that the first fuse is defective, $B$ is the event that the second fuse is defective, and $C$ is the event that the third fuse is defective, then

$$P(A) = \frac{5}{20}, \; P(B|A) = \frac{4}{19}, \text{ and } P(C|A \cap B) = \frac{3}{18},$$

and substitution into the formula yields

$$\begin{aligned} P(A \cap B \cap C) &= P(A) \cdot P(B|A) \cdot P(C|A \cap B) \\ &= \frac{5}{20} \cdot \frac{4}{19} \cdot \frac{3}{18} \\ &= \frac{1}{114}. \end{aligned}$$

# Outline

# Independent Events

If $P(B|A) = P(B)$, i.e., the probability of $B$ occurring is not affected by the occurrence or non-occurrence of $A$, then we say that $A$ and $B$ are *independent events*. This is equivalent to

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B|A) \\ &= P(A) \cdot P(B). \end{aligned}$$

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**Definition 28**

Two events $A$ and $B$ are *independent* if and only if

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**Definition 28**

Two events $A$ and $B$ are *independent* if and only if

$$P(A \cap B) = P(A) \cdot P(B).$$

If two events are not independent, they are said to be *dependent*.

## Example 29

A coin is tosses three times and the eight possible outcomes, $HHH, HHT, HTH, THH, HTT, THT, TTH$ and $TTT$, are assumed to be equally likely. If $A$ is the event that a head occurs on each of the first two tosses, $B$ is the event that a tail occurs on the third toss, and $C$ is the event that exactly two tails occur in the three tosses, show that **(a)** events $A$ and $B$ are independent **(b)** events $B$ and $C$ are dependent.

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**Solution.** Since

$$A = \{HHH, HHT\},\ B = \{HHT, HTT, THT, TTT\},$$

$$C = \{HTT, THT, TTH\},\ A \cap B = \{HHT\}, B \cap C = \{HTT, THT\}$$

the assumption that the eight possible outcomes are all equiprobable yields

$$P(A) = \frac{2}{8},\ P(B) = \frac{4}{8}, P(C) = \frac{3}{8},\ P(A \cap B) = \frac{1}{8}, P(B \cap C) = \frac{2}{8}.$$

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**(a)** Since

$$P(A) \cdot P(B) = \frac{2}{8} \cdot \frac{4}{8} = \frac{1}{8} = P(A \cap B),$$

the events $A$ and $B$ are independent.

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**(a)** Since

$$P(A) \cdot P(B) = \frac{2}{8} \cdot \frac{4}{8} = \frac{1}{8} = P(A \cap B),$$

the events $A$ and $B$ are independent.
**(b)** Since

$$P(B) \cdot P(C) = \frac{4}{8} \cdot \frac{3}{8} = \frac{3}{16} \neq \frac{2}{8} = P(B \cap C),$$

the events $B$ and $C$ are dependent.

## Example 30

In the Yahtzee dice game 5 dice are rolled and a winning roll has three of a king and two of a kind (but not five of a kind). **(a)** Compute the probability of winning. **(b)** What is the probability of not getting a single winning roll in 5 tries? **(c)** What is the probability of getting 3 winners in 44 tries?

## Example 30

In the Yahtzee dice game 5 dice are rolled and a winning roll has three of a king and two of a kind (but not five of a kind). **(a)** Compute the probability of winning. **(b)** What is the probability of not getting a single winning roll in 5 tries? **(c)** What is the probability of getting 3 winners in 44 tries?

**Solution.** **(a)** $P(win) = \frac{\binom{5}{3}\cdot\binom{2}{2}\cdot 6\cdot 5}{6\cdot 6\cdot 6\cdot 6\cdot 6} = \frac{10\cdot 1\cdot 6\cdot 5}{6^5} = 0.0386.$

## Example 30

In the Yahtzee dice game 5 dice are rolled and a winning roll has three of a king and two of a kind (but not five of a kind). **(a)** Compute the probability of winning. **(b)** What is the probability of not getting a single winning roll in 5 tries? **(c)** What is the probability of getting 3 winners in 44 tries?

**Solution.** **(a)** $P(win) = \frac{\binom{5}{3}\cdot\binom{2}{2}\cdot 6\cdot 5}{6\cdot 6\cdot 6\cdot 6\cdot 6} = \frac{10\cdot 1\cdot 6\cdot 5}{6^5} = 0.0386.$
**(b)** Each roll of the 5 dice together is one try. Each try is independent of the other tries. Therefore

$$\begin{aligned} P(lose \cap lose \cap lose \cap lose \cap lose) &= P(lose)\cdot P(lose)\cdots P(lose) \\ &= (P(lose))^5 = (1 - P(win))^5 \\ &= (1 - 0.0386)^5 = 0.82. \end{aligned}$$

**(c)** Let consider the sequence *win*, *win*, *win*, followed by $44 - 3 = 41$ *loses*. Then, with this order we have

$$P(3win, 41lose \text{ with order}) = (P(win))^3 \cdot (P(lose))^{41}$$
$$= 0.0386^3 \cdot (1 - 0.0386)^{41} = 1.14 \times 10^{-5}.$$

But there are $\binom{44}{3}$ ways to choose three winning rolls in 44 tries. So,

$$P(3win, 41lose \text{ without order}) = \binom{44}{3} \cdot 1.14 \times 10^{-5}$$
$$= 0.15.$$

**Theorem 31**

*If $A$ and $B$ are independent, then $A$ and $B'$ are also independent.*

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**Proof.**

Since $A = (A \cap B) \cup (A \cap B')$, $A \cap B$ and $A \cap B'$ are mutually exclusive, and $A$ and $B$ are independent by assumption, we have

$$\begin{aligned} P(A) &= P[(A \cap B) \cup (A \cap B')] \\ &= P(A \cap B) + P(A \cap B') \\ &= P(A) \cdot P(B) + P(A \cap B'). \end{aligned}$$

It follows that

$$\begin{aligned} P(A \cap B') &= P(A) - P(A) \cdot P(B) \\ &= P(A)[1 - P(B)] \\ &= P(A) \cdot P(B') \end{aligned}$$

and hence that $A$ and $B'$ are independent. □

## Example 32

Find the probabilities of getting **(a)** three heads in three random tosses of a balanced coin; **(b)** four sixes and then another number in five random rolls of a balanced die.

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**Solution. (a)** Multiplying the respective probabilities, we get

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

## Example 32

Find the probabilities of getting **(a)** three heads in three random tosses of a balanced coin; **(b)** four sixes and then another number in five random rolls of a balanced die.

**Solution. (a)** Multiplying the respective probabilities, we get

$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$$

**(b)** Multiplying the respective probabilities, we get

$$\frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} = \frac{5}{7,776}.$$

# Outline

# Bayes' Theorem

In many situations the outcome of an experiment depends on what happens in various intermediate stages.

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## Example 33

The completion of a construction job may be delayed because of a strike. The probabilities are 0.60 that there will be a strike, 0.85 that the construct job will be completed on time if there is no strike, and 0.35 that the construction job will be completed on time if there is a strike. What is the probability that the construction job will be completed on time?

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**Solution.** If $A$ is the event that the construction job will be completed on time and $B$ is the event that there will be a strike, we are given $P(B) = 0.6$, $P(A|B') = 0.85$, and $P(A|B) = 0.35$.

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**Solution.** If $A$ is the event that the construction job will be completed on time and $B$ is the event that there will be a strike, we are given $P(B) = 0.6$, $P(A|B') = 0.85$, and $P(A|B) = 0.35$. Since $A \cap B$ and $A \cap B'$ are mutually exclusive,

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**Solution.** If $A$ is the event that the construction job will be completed on time and $B$ is the event that there will be a strike, we are given $P(B) = 0.6$, $P(A|B') = 0.85$, and $P(A|B) = 0.35$. Since $A \cap B$ and $A \cap B'$ are mutually exclusive, then

$$\begin{aligned} P(A) &= P[(A \cap B) \cup (A \cap B')] \\ &= P(A \cap B) + P(A \cap B') \\ &= P(B) \cdot P(A|B) + P(B') \cdot P(A|B') \\ &= (0.6)(0.35) + (1 - 0.6)(0.85) \\ &= 0.55. \end{aligned}$$

An immediate generalization of this kind of situation is the case where the intermediate stage permits $k$ different alternatives (whose occurrence is denoted by $B_1, B_2, \ldots, B_k$). It requires the following theorem, sometimes called the *rule of total probability* or *rule of elimination*.

An immediate generalization of this kind of situation is the case where the intermediate stage permits $k$ different alternatives (whose occurrence is denoted by $B_1, B_2, \ldots, B_k$). It requires the following theorem, sometimes called the *rule of total probability* or *rule of elimination*.

**Theorem 34**

*If the events $B_1, B_2, \ldots,$ and $B_k$ constitute a partition of the sample space $\mathcal{S}$ and $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event $A$ in $\mathcal{S}$*

$$P(A) = \sum_{i=1}^{k} P(B_i) \cdot P(A|B_i).$$

## Example 35

The members of a consulting firm rent cars from three rental agencies: 60 percent from agency 1, 30 percent from agency 2, and 10 percent from agency 3. If 9 percent of the cars from agency 1 need a tune-up, 20 percent of the cars from agency 2 need a tune-up, and 6 percent of the cars from agency 3 need a tune-up, what is the probability that a rental car delivered to the firm will need a tune-up?

## Example 35

The members of a consulting firm rent cars from three rental agencies: 60 percent from agency 1, 30 percent from agency 2, and 10 percent from agency 3. If 9 percent of the cars from agency 1 need a tune-up, 20 percent of the cars from agency 2 need a tune-up, and 6 percent of the cars from agency 3 need a tune-up, what is the probability that a rental car delivered to the firm will need a tune-up?

**Solution.** If $A$ is the event that the car needs a tune-up, and $B_1$, $B_2$, and $B_3$ are the events that the car comes from rental agencies 1, 2, or 3,

## Example 35

The members of a consulting firm rent cars from three rental agencies: 60 percent from agency 1, 30 percent from agency 2, and 10 percent from agency 3. If 9 percent of the cars from agency 1 need a tune-up, 20 percent of the cars from agency 2 need a tune-up, and 6 percent of the cars from agency 3 need a tune-up, what is the probability that a rental car delivered to the firm will need a tune-up?

**Solution.** If $A$ is the event that the car needs a tune-up, and $B_1$, $B_2$, and $B_3$ are the events that the car comes from rental agencies 1, 2, or 3, we have $P(B_1) = 0.60$, $P(B_2) = 0.30$, $P(B_3) = 0.10$, $P(A|B_1) = 0.09$, $P(A|B_2) = 0.20$, and $P(A|B_3) = 0.06$. Thus,

$$\begin{aligned} P(A) &= P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + P(B_3) \cdot P(A|B_3) \\ &= (0.60)(0.09) + (0.30)(0.20) + (0.10)(0.06) \\ &= 0.12. \end{aligned}$$

With reference to the preceding example, suppose that we are interested in the following question:

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With reference to the preceding example, suppose that we are interested in the following question: If a rental car delivered to the consulting firm needs a tune-up, what is the probability that it came from rental agency 2? To answer questions of this kind, we need the following theorem, called **Bayes' Theorem**:

**Theorem 36**

*If $B_1, B_2, \cdots, B_k$ constitute a partition of a sample space $\mathcal{S}$ and $P(B_i) \neq 0$ for $i = 1, 2, \cdots, k$, then for any event $A$ in $\mathcal{S}$ such that $P(A) \neq 0$*

$$P(B_r|A) = \frac{P(B_r) \cdot P(A|B_r)}{\sum_{i=1}^{k} P(B_i) \cdot P(A|B_i)}$$

*for $r = 1, 2, \cdots, k$.*

## Example 37

With reference to Example 35, if a rental car delivered to the consulting firm needs a tune-up, what is the probability that it came from rental agency 2?

## Example 37

With reference to Example 35, if a rental car delivered to the consulting firm needs a tune-up, what is the probability that it came from rental agency 2?

**Solution.**

$$
\begin{aligned}
P(B_2|A) &= \frac{P(B_2)\cdot P(A|B_2)}{P(B_1)\cdot P(A|B_1) + P(B_2)\cdot P(A|B_2) + P(B_3)\cdot P(A|B_3)} \\
&= \frac{(0.30)(0.20)}{(0.60)(0.09) + (0.30)(0.20) + (0.10)(0.06)} \\
&= \frac{0.060}{0.120} = 0.5
\end{aligned}
$$

Observe that although only 30 percent of the cars delivered to the firm come from agency 2, 50 percent of those requiring a tune-up come from that agency.

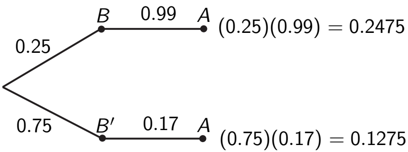
## Example 38

In a certain state, 25 percent of all cars emit excessive amounts of pollutants. If the probability is 0.99 that a car emitting excessive amounts of pollutants will fail the state's vehicular emission test, and the probability is 0.17 that a car not emitting excessive amounts of pollutants will nevertheless fail the test, what is the probability that a car that fails the test actually emits excessive amounts of pollutants?

## Example 38

In a certain state, 25 percent of all cars emit excessive amounts of pollutants. If the probability is 0.99 that a car emitting excessive amounts of pollutants will fail the state's vehicular emission test, and the probability is 0.17 that a car not emitting excessive amounts of pollutants will nevertheless fail the test, what is the probability that a car that fails the test actually emits excessive amounts of pollutants?

**Solution.** Picturing this situation as in the figure




we find the probabilities associated with the two branches of the tree diagram are $(0.25)(0.99) = 0.2475$ and $(1 - 0.25)(0.17) = 0.1275$.

Thus, the probability that a car that fails the test actually emits excessive amounts of pollutants is

$$\frac{0.2475}{0.2475 + 0.1275} = 0.66.$$

Of course, this result could also have been obtained without the diagram by substituting directly into the formula of Bayes' theorem.

## Example 39

While watching a game of Champions League football in a cafe, you observe someone who is clearly supporting Manchester United in the game. What is the probability that they were actually born within 25 miles of Manchester? Assume that:

- the probability that a randomly selected person in a typical local bar environment is born within 25 miles of Manchester is $1/20$, and;
- the chance that a person born within 25 miles of Manchester actually supports United is $7/10$;
- the probability that a person not born within 25 miles of Manchester supports United with probability $1/10$.

**Solution.** If $B$ is the event the person is born within 25 miles of Manchester, and $A$ is the event that the person supports United, we have

**Solution.** If $B$ is the event the person is born within 25 miles of Manchester, and $A$ is the event that the person supports United, we have

$$P(B) = \frac{1}{20}, \; P(A|B) = \frac{7}{10}, \; P(A|B') = \frac{1}{10}.$$

**Solution.** If $B$ is the event the person is born within 25 miles of Manchester, and $A$ is the event that the person supports United, we have

$$P(B) = \frac{1}{20},\ P(A|B) = \frac{7}{10},\ P(A|B') = \frac{1}{10}.$$

By Bayes's Theorem,

$$\begin{aligned} P(B|A) =& \frac{P(B)P(A|B)}{P(B)P(A|B) + P(B')P(A|B')} \\ =& \frac{\frac{1}{20}\frac{7}{10}}{\frac{1}{20}\frac{7}{10} + \left(1 - \frac{1}{20}\right)\frac{1}{10}} \\ =& \frac{7}{26}. \end{aligned}$$

## Example 40

Suppose that a drug test for an illegal drug is such that it is 98% accurate in the case of a user of that drug (e.g. it produces a positive result with probability 0.98 in the case that the tested individual uses the drug) and 90% accurate in the case of a non-user of the drug (e.g. it is negative with probability 0.90 in the case the person does not use the drug). Suppose it is known that 10% of the entire population uses this drug.

**(a)** You test someone and the test is positive. What is the probability that the tested individual uses this illegal drug?

**(b)** What is the probability of a false positive with this test (e.g. the probability of obtaining a positive drug test given that the person tested is a non-user)?

**(c)** What is the probability of obtaining a false negative for this test (e.g. the probability that the test is negative, but the individual tested is a user)?

Let $+$ be the event that the drug test is positive for an individual, $-$ be the event that the drug test is negative for an individual, and $A$ be the event that the person tested does use the drug that is being tested for.

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(a)

$$\begin{aligned} P(A|+) &= \frac{P(A)P(+|A)}{P(A)P(+|A) + P(A')P(+|A')} \\ &= \frac{(0.10)(0.98)}{(0.10)(0.98) + (0.90)(0.10)} \\ &= 0.52 \end{aligned}$$

(b) The probability of a false positive is

$$\begin{aligned} P(+|A') &= 1 - P(-|A') \\ &= 1 - 0.90 \\ &= 0.10 \end{aligned}$$

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(c) The probability of a false negative is

$$\begin{aligned} P(-|A) &= 1 - P(+|A) \\ &= 1 - 0.98 \\ &= 0.02. \end{aligned}$$

# The Monty Hall Problem

The Monty Hall problem is a probability puzzle loosely based on the American television game show Let's Make a Deal and named after the show's original host, Monty Hall. The problem, also called the Monty Hall paradox, is a veridical paradox because the result appears impossible but is demonstrably true.

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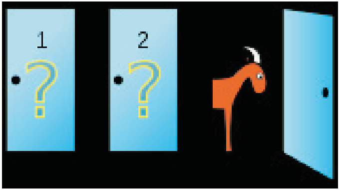
**Example 41**

Suppose you are on a game show, and you are given the choice of three doors: Behind one door is a car; behind the others, goats. You pick a door, say No. 1 [but the door is not opened], and the host, who knows what is behind the doors, opens another door, say No. 3, which has a goat. He then says to you, *Do you want to pick door No. 2*? Is it to your advantage to switch your choice?




In search of a new car, the player picks a door, say 1. The game host then opens one of the other doors, say 3, to reveal a goat and offers to let the player pick door 2 instead of door 1.

# vos Savant's Solution

There are three possible arrangements of one car and two goats behind three doors and the result of staying or switching after initially picking door 1 in each case:

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| behind door 1 | behind door 2 | behind door 3 | result if staying at door 1 | result if switching to the door offered |
|---|---|---|---|---|
| **Car** | Goat | Goat | **Car** | Goat |
| Goat | **Car** | Goat | Goat | **Car** |
| Goat | Goat | **Car** | Goat | **Car** |

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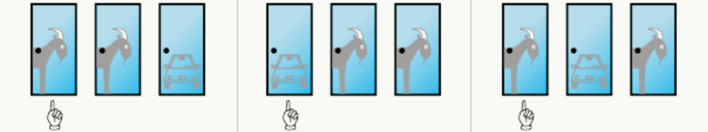
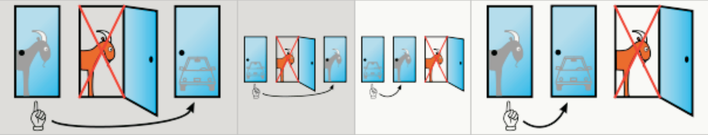
| Goat | Goat | **Car** | Goat | **Car** |

A player who stays with the initial choice wins in only one out of three of these equally likely possibilities, while a player who switches wins in two out of three. The probability of winning by staying with the initial choice is therefore $1/3$, while the probability of winning by switching is $2/3$.

# Another Simple Solution

| Car hidden behind door 3 | Car hidden behind door 1 | | Car hidden behind door 2 |
|---|---|---|---|
| Player initially picks door 1 | | | |
|  | | | |
| Host must open door 2 | Host randomly opens door 2 | Host randomly opens door 3 | Host must open door 3 |
|  | | | |
| Probability 1/3 | Probability 1/6 | Probability 1/6 | Probability 1/3 |
| Switching wins | Switching loses | Switching loses | Switching wins |
| If the host has opened door 3, these cases have not happened | | If the host has opened door 3, switching wins twice as often as staying | |

# Solution of the Monty Hall Problem using the Bayes' Theorem

Let $\{1, 2, 3\}$ be the set of door numbers, $C$ be the event that the number of the door hiding the **C**ar, $S$ be the event that the number of the door **S**elected by the player, and $H$ be the event that the number of the door opened by the **H**ost.

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$$P(C = 1|S = 1) = P(C = 2|S = 1) = P(C = 3|S = 1) = \frac{1}{3},$$

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$$P(C = 1|S = 1) = P(C = 2|S = 1) = P(C = 3|S = 1) = \frac{1}{3},$$

and due to the host's behavior:

$$P(H = 3|C = 1, S = 1) = \frac{1}{2}, P(H = 3|C = 2, S = 1) = 1,$$

$$P(H = 3|C = 3, S = 1) = 0.$$

If we let

$$p_1 := P(H = 3|C = 1, S = 1)P(C = 1|S = 1) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6},$$

$$p_2 := P(H = 3|C = 2, S = 1)P(C = 2|S = 1) = 1 \cdot \frac{1}{3} = \frac{1}{3},$$

and

$$p_3 := P(H = 3|C = 3, S = 1)P(C = 3|S = 1) = 0 \cdot \frac{1}{3} = 0,$$

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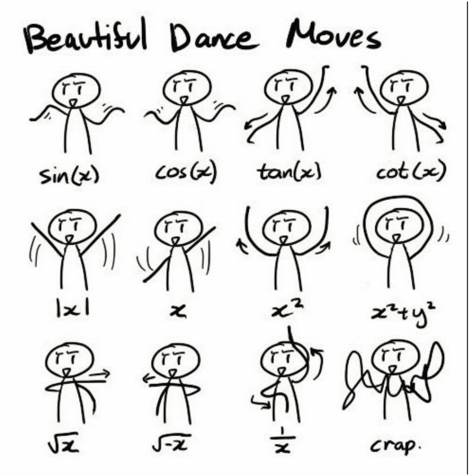
$$p_3 := P(H = 3|C = 3, S = 1)P(C = 3|S = 1) = 0 \cdot \frac{1}{3} = 0,$$

so, the probability of winning by switching is

$$\begin{aligned} P(C = 2|H = 3, S = 1) &= \frac{p_2}{p_1 + p_2 + p_3} \\ &= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3}} \\ &= \frac{2}{3}. \end{aligned}$$


Beautiful Dance Moves
sin(x)
cos(x)
tan(x)
cot(x)
|x|
x
x²
x²+y²
√x
√-x
1/x
crap.

Thank You!!!